Finding Equations of Motion

with Maple

http://wbrenna.uwaterloo.ca/wilson/projects/maple/ibp.m

http://wbrenna.uwaterloo.ca/wilson/projects/maple/physicsdiff.m

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Integration by Parts

$$f(x)\,g(x)\Big|_{x=a}^{x=b} \overset{\text{fundamental th}^{m}\text{ of calculus}}{=\!=\!=} \int_a^b \frac{d}{dx}\left(f(x)\,g(x)\right)dx$$

Expand:

$$f(x)\,g(x)\Big|_{x=a}^{x=b} = \int_a^b f'(x)\,g(x)\,dx + \int_a^b f(x)\,g'(x)\,dx$$

$$\int_a^b f'(x)\,g(x)\,dx = f(x)\,g(x)\Big|_{x=a}^{x=b} - \int_a^b f(x)\,g'(x)\,dx$$

Maple ~> *IntegrationTools[Parts]:* *Parts(integral,function);*

Maple Package: *ibp.m*

Mac OS X:

1. *Find Maple install folder. This might be "/Applications/Maple13"*
2. *Move the ibp.m into this folder/a subfolder.*
3. *> read `/Applications/Maple13/ibp.m`*

↑ note backticks

Windows:

1. *Find Maple.*
   XP: "C:\Documents and Settings\<username>\Application Data\Maple\"
   Vista: "C:\Users\<username>\AppData\Roaming\Maple\"
   or "C:\Program Files\Maple XX\"
2. *Move ibp.m to some subfolder of that folder.*
3. *>read `C:\\Users\\<u.n.>\\AppData\\Roaming\\Maple\\ibp.m`*

← this is different on every machine I've seen!

↑ need double-backslash

Linux:

1. *Put ibp.m anywhere permanent on your system.*
2. *>read `/path/to/ibp.m`*

Autoloading

Mac OS X: *Go to the Maple folder and find the file "MapleInit". If it's not there, create a plain text file with the same name. Add the line:*

```
read("/Applications/MapleXX/subfolder/ibp.m"):
```

Windows: *Go to the Maple folder. Find the file "maple.ini" in the "lib" or "Users" or "bin" subfolder. Add the line:*

```
read("C:\\path\\to\\ibp.m"):
```

Linux: *Create/edit the file "/home/<username>/.mapleinit". Add the line:*

```
read("/path/to/ibp.m"):
```

Epic Autoloading

```
files := map(FileTools:=AbsolutePath,FileTools:-ListDirectory
                ("/home/wilson/.maple/permafolder",'returnonly' = "*.m"),
                "/home/wilson/.maple/permafolder"):
for file in files do read file; end do;
```

I may need backticks for Windows ?

Ok. IBP Usage > *Describe(ibp);*

> *ibp(expression, power, diff-variable, ([functions], tryhard));*

IBP tries to eliminate a given order of derivative w.r.t. diff-variable.

optional: only integrate out derivatives of these functions

*note: must be a list

for maximum power. Sometimes this will return a correct but messy result.

Set to true/false.

Notes:

* never any boundary terms
* often works with recursion:

ibp(expr) = K * expr + terms

e.g. Put

$$\mathcal{S}' = \int dr \left[-\partial_r \phi_{(r)} \cdot \partial_r \phi_{(r)} - m^2 \phi^2_{(r)} \right] \cdot \frac{1}{2}$$

in the form of $\mathcal{S}'[\phi_{(r)}, \partial_r^2 \phi_{(r)}]$.

Soln: Integrate the 1st term by parts.

$$\mathcal{S}' = \int dr \left[+\left(\partial_r^2 \phi_{(r)} \right) \phi_{(r)} - m^2 \phi^2_{(r)} \right] \cdot \frac{1}{2}$$

```
> expr := 1/2*( -diff(phi(r),r)^2 - m^2*phi(r)^2 );
> ibp( expr, 1, r, [phi(r)], true );
```

Maple Package: *physicsdiff.m* physicsdiff

Mac OS X:

1. *Find the Maple install folder. This might be "/Applications/Maple13"*
2. *Move the physicsdiff.m into this folder/a subfolder.*
3. *> read `/Applications/Maple13/physicsdiff.m`*

Windows:

1. *Find Maple.*
   XP: "C:\Documents and Settings\<username>\Application Data\Maple\"
   Vista: "C:\Users\<username>\AppData\Roaming\Maple\" etc. etc.
2. *Move physicsdiff.m to some subfolder of that folder.*
3. *>read `C:\\Users\\<u.n.>\\AppData\\Roaming\\Maple\\physicsdiff.m`*

Linux:

1. *Put physicsdiff.m anywhere permanent on your system.*
2. *>read `/path/to/physicsdiff.m`*

Physicsdiff Usage

> *Describe(physicsdiff);*

> *physicsdiff(expression, function, diff-variable);*

* in Beta!

vary the expression w.r.t. function

pass as list

the variable [s] that function depends on

Notes:

- ✱ never any boundary terms
- ✱ attempts to integrate by parts (weaker than IBP)

e.g. Put $\mu = [r, t]$

$$\mathcal{S} = \int dt\, dr \left[\partial_\mu \phi_{(r,t)} \cdot \partial^\mu \phi_{(r,t)} - m^2 \phi^2_{(r,t)} - \frac{k \phi^4_{(r,t)}}{12} \right] \cdot \frac{1}{2}$$

in the form of $\mathcal{S}[\phi_{(r,t)}, \partial^2_\mu \phi_{(r,t)}]$.

Soln: Vary w.r.t. $\phi_{(r,t)}$. Then, integrate the 1st term by parts so we can extract $\delta \phi_{(r,t)}$.

$$\delta \mathcal{S} = \int dt\, dr \left[\partial_\mu \phi_{(r,t)} \partial^\mu \delta \phi_{(r,t)} + \partial_\mu \phi_{(r,t)} \partial^\mu \delta \phi_{(r,t)} - 2 m^2 \phi_{(r,t)} \delta \phi_{(r,t)} - \delta \phi_{(r,t)} k \phi^3_{(r,t)} / 3 \right] \cdot \frac{1}{2}$$

$$= \int dt\, dr\, \delta \phi_{(r,t)} \left[-\partial^\mu \partial_\mu \phi_{(r,t)} - m^2 \phi_{(r,t)} - k \phi^3_{(r,t)} / 6 \right]$$

So our equation of motion is

$$\partial_r^2 \phi - \partial_t^2 \phi - m^2 \phi - \frac{k}{6} \phi^3 = 0$$

Now, turn to Maple ...

```
expr := 1/2*(diff(phi(t,r),t)^2 - diff(phi(t,r),r)^2 - m^2*phi(t,r)^2 - k/12*phi(t,r)^4);
physicsdiff(expr, phi(t,r), [r,t] );
```

E.g. A more challenging example.

$$S = \int dr \left[3 \cdot g'(r) \, f(r) / \sqrt{f(r) g(r)} \right]$$

Two field equations: $\frac{\delta S}{\delta f(r)} = \frac{1}{2} \cdot \frac{1}{\sqrt{f(r)}} \cdot \frac{3 g'(r)}{\sqrt{g(r)}}$

IBP to get $S[f'(r), g(r), f(r)]$ for the 2nd field eqn.

$$S = -\int dr \cdot 3 g(r) \frac{d}{dr} \left[f(r) / \sqrt{f(r) g(r)} \right]$$

original term / 2

$$= -\int dr \cdot 3 g(r) \left[\frac{1}{2} \cdot \frac{f'(r)}{\sqrt{f(r)}} \cdot \frac{1}{\sqrt{g(r)}} - \frac{1}{2} \frac{g'(r)}{(g(r))^{3/2}} \sqrt{f(r)} \right]$$

$$= -\int dr \cdot 3g(r) \left[\sum_{n=1}^{\infty} \frac{1}{(2)^n} \left\{ \frac{f'(r)}{\sqrt{f(r)}} \cdot \frac{1}{\sqrt{g(r)}} \right\} - \frac{1}{(\pm 2)^{\infty}} \frac{g'(r)}{(g(r))^{3/2}} \sqrt{f(r)} \right]$$

$$= -\int dr \cdot 3g(r) \left[\frac{f'(r)}{\sqrt{f(r)}} \cdot \frac{1}{\sqrt{g(r)}} \right]$$

$$\text{or} \quad g(r)' \, f(r) / \sqrt{f(r) g(r)} = \left[\frac{-1}{2} \frac{f'(r)}{\sqrt{f(r)}} \cdot \frac{1}{\sqrt{g(r)}} + \frac{1}{2} \frac{g'(r)}{g(r)^{3/2}} \sqrt{f(r)} \right] g(r)$$

$$\Rightarrow \frac{1}{2} g(r)' \, f(r) / \sqrt{f(r) g(r)} = \frac{-1}{2} \frac{f'(r)}{\sqrt{f(r)}} \cdot \frac{1}{\sqrt{g(r)}}$$

$$g(r)' \, f(r) / \sqrt{f(r) g(r)} = -\frac{f'(r)}{\sqrt{f(r)}} \cdot \frac{1}{\sqrt{g(r)}}$$

Then, $\dfrac{\delta \mathcal{S}}{\delta g_{(r)}} = -\dfrac{3}{2}\dfrac{f'_{(r)}}{\sqrt{f_{(r)}}}\cdot\dfrac{1}{\sqrt{g_{(r)}}}$

[N.B. $\delta \mathcal{S} \equiv \int \dfrac{\delta \mathcal{S}}{\delta f_{(x)}} \delta f_{(x)}\, dx + \int \dfrac{\delta \mathcal{S}}{\delta g_{(x)}} \delta g_{(x)}\, dx$]

$$\int \frac{\delta \mathcal{S}}{\delta f_{(x)}} d_{(x)}\, dx = \frac{d}{d\varepsilon} \mathcal{S}\left[f_{(x)} + \varepsilon d_{(x)} \right]\Big|_{\varepsilon=0}$$

$d = \delta_{(x)}$ gives us the way we 'apply' δ on $\mathcal{S}$.

Contest

(1) I choose expr.

(2) You choose expr.

- → only one-dimensional integrals
- → 2nd order derivatives at most
- → arbitrary powers of functions allowed!
- → please don't make it trivial :)

(3) We each choose an expr. and add them.